Game Theory

Chapter 2
Solution Methods for Matrix Games

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Contents

Solution of Some Special Games

2×2 Games Revisited

- We have seen that any 2 x 2 matrix game can be solved graphically. There are also explicit formulas giving the value and optimal strategies with the advantage that they can be run on a calculator or computer.

- Consider the game with matrix and strategies

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{player I} : X = (x, 1-x); \quad \text{player II} : Y = (y, 1-y).$$

$$E(X,Y) = XAY^T$$

$$= xy(a_{11} - a_{12} - a_{21} + a_{22}) + x(a_{12} - a_{22}) + y(a_{21} - a_{22}) + a_{22}.$$

Formulas

- **Theorem 2.1.1** *In the* 2×2 *game with matrix* A, **assume that there are no pure optimal strategies**. *If we set*

$$x^* = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}, \qquad y^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}},$$

then $X^* = (x^*, 1 - x^*), Y^* = (y^*, 1 - y^*)$ *are optimal mixed strategies for players I and II, respectively. The value of the game is*

$$v(A) = E(X^*, Y^*) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}.$$

Formulas (cont'd)

- Derivation of formulas

$f(x,y) = E(X,Y),\ X = (x, 1-x), Y = (y, 1-y), 0 \le x, y \le 1$. Then

$$f(x,y) = (x, 1-x) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} \tag{2.1.1}$$
$$= x[y(a_{11} - a_{21}) + (1-y)(a_{12} - a_{22})] + a_{12} - a_{22}.$$

Formulas (cont'd)

– Let $\frac{\partial f}{\partial x} = y\alpha + \beta = 0$ and $\frac{\partial f}{\partial y} = x\alpha + \gamma = 0,$

where $\alpha = (a_{11} - a_{12} - a_{21} + a_{22}),\ \ \beta = (a_{12} - a_{22}),\ \ \gamma = (a_{21} - a_{22}).$

– Notice that if $\alpha = 0$, the partials are never zero (assuming $\beta, \gamma \neq 0$), and that would imply that there are pure optimal strategies (in other words, the min and max must be on the boundary).
– Solve where the partial derivatives are zero to get .

$$x^* = -\frac{\gamma}{\alpha} = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} \quad \text{and} \quad y^* = -\frac{\beta}{\alpha} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

Figure 2.1 is the graph of $f(x, y) = XAY^T$ taking $A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$.

Formulas (cont'd)

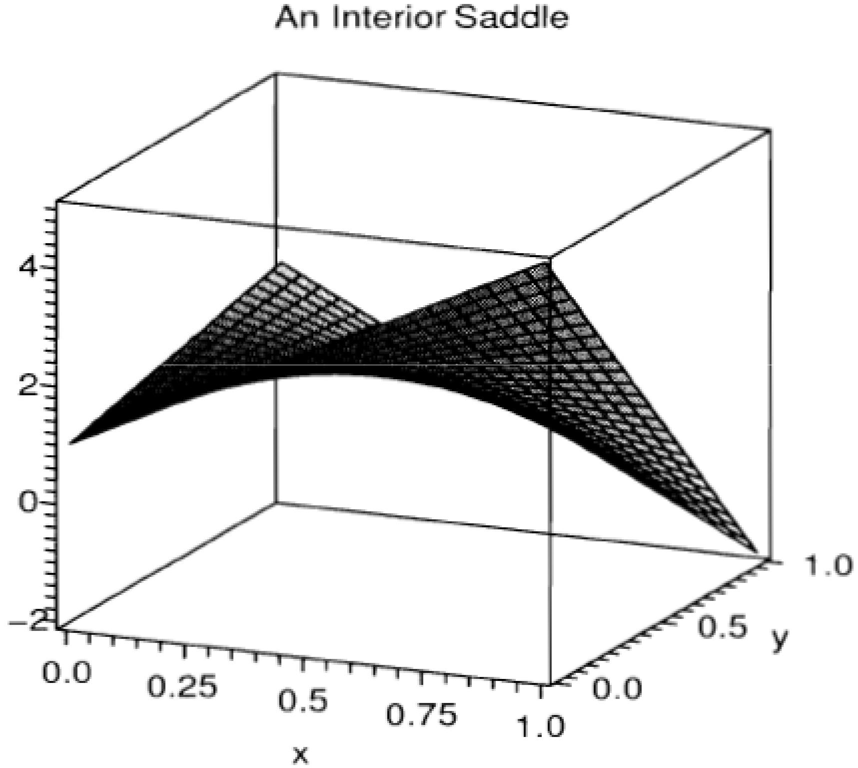

Figure 2.1 Concave in x, convex in y, saddle at $(\frac{1}{8}, \frac{1}{2})$.

Formulas (cont'd)

– This is <u>a saddle and not a min or max of *f*.</u> The reason is because if we take the second derivatives, we get the matrix of second partials (called the **Hessian**):

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$$

Since $\det(H) = -\alpha^2 < 0$ (unless $\alpha = 0$, which is ruled out) a theorem in elementary calculus says that an interior critical point with this condition must be a saddle.

- The calculus definition of a saddle point of a function *f*(x, *y)* is a point so that in every neighborhood of the point there are *x* and *y* values that make *f* bigger and smaller than *f* at the candidate saddle point.

Formulas (cont'd)

- Remarks
  - The main assumption you need before you can use the formulas is that the game does **not** have a pure saddle point.
    - Check whether $a_{11} - a_{12} - a_{21} + a_{22} = 0,\ v^{+} = v^{-}$.
  - A more compact way to write the formulas and easier to remember is

$$X^* = \frac{(1\ 1)A^*}{(1\ 1)A^* \begin{bmatrix}1\\1\end{bmatrix}} \quad \text{and} \quad Y^* = \frac{A^* \begin{bmatrix}1\\1\end{bmatrix}}{(1\ 1)A^* \begin{bmatrix}1\\1\end{bmatrix}}$$

$$value(A) = \frac{\det(A)}{(1\ 1)A^* \begin{bmatrix}1\\1\end{bmatrix}}$$

where

$$A^* = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \text{ and } \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Example 2.1

- In the game with $A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$, $v^- = 1 < v^+ = 2$
  - There is no pure saddle.
  - Apply the formulas to get

    $X^* = (\frac{1}{8}, \frac{7}{8})$ and $Y^* = (\frac{1}{2}, \frac{1}{2})$, $v(A) = \frac{3}{2}$.

  - Notice here that

    $$A^* = \begin{bmatrix} 1 & -5 \\ -2 & -2 \end{bmatrix}, \ \det(A) = -12.$$

    The matrix formula for player I gives

    $$X = \frac{(1\ 1)A^*}{((1\ 1)A^*(1\ 1)^T)} = \left(\frac{1}{8}, \frac{7}{8}\right).$$

Invertible Matrix Games

Invertible Matrix Games

- **Theorem 2.2.1** *Assume that*
  1. $A_{n\times n}$ *has an inverse* A^{-1}.
  2. $J_n A^{-1} J_n^T \neq 0$. $J_n = (1\ 1\ \cdots 1)$
  3. $v(A) \neq 0$.

  Set $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_m)$, *and*

$$v \equiv \frac{1}{J_n A^{-1} J_n^T}, \qquad Y^T = \frac{A^{-1} J_n^T}{J_n A^{-1} J_n^T}, \qquad X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

  If $x_i \geq 0, i = 1, \ldots, n$ *and* $y_j \geq 0, j = 1, \ldots, n$, *we have that* $v = v(A)$ *is the value of the game with matrix* A *and* (X, Y) *is a saddle point in mixed strategies.*

Invertible Matrix Games (cont'd)

- Formulation of Theorem 2.2.1
  - Suppose that player I has an optimal strategy that is completely mixed. By the Properties of Strategies (1.3.1), property 3, every optimal *Y* strategy for player II, must satisfy

$$E(i, Y) = {}_iAY^T = value(A), \text{ for every row } i = 1, 2, \ldots, n.$$

  - If we write $J_n = (1\ 1\ \cdots 1)$ for the row vector consisting of all Is, we can write

$$AY^T = v(A)J_n^T = \begin{bmatrix} v(A) \\ \vdots \\ v(A) \end{bmatrix}. \qquad (2.2.1)$$

Invertible Matrix Games (cont'd)

- The value of the game cannot be zero.
- Now, if $v(A) = 0$, then $AY^T = 0J_n^T = 0$. ➔ $Y = A^{-1}0 = 0$, where $Y = (y_1, \ldots, y_n)$.

  But that is impossible if Y=0 is a strategy (the components must add to 1).

Invertible Matrix Games (cont'd)

– Multiply both sides of (2.2.1) by A^{-1}, we get

$$A^{-1}AY^T = Y^T = v(A)A^{-1}J_n^T.$$

This gives us *Y* if we knew *v*(*A*).

– With the extra piece of information $\sum_{j=1}^{n} y_j = J_n Y^T = 1$, we get

$$J_n Y^T = 1 = v(A) J_n A^{-1} J_n^T,$$

and therefore

$$v(A) = \frac{1}{J_n A^{-1} J_n^T} \text{ and then } Y^T = \frac{A^{-1} J_n^T}{J_n A^{-1} J_n^T}.$$

Invertible Matrix Games (cont'd)

- Verification
  - Let $Y' \in S_n$ be any mixed strategy and let *X* be given by the formula $X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}$. Then, since $J_n Y'^T = 1$, we have

$$E(X, Y') = XAY'^T = \frac{1}{J_n A^{-1} J_n^T} J_n A^{-1} A Y'^T$$
$$= \frac{1}{J_n A^{-1} J_n^T} J_n Y'^T$$
$$= \frac{1}{J_n A^{-1} J_n^T} = v.$$

  - Similarly, for any $X' \in S_n,\ E(X', Y) = v$.
  - So (X, Y) is a saddle and *v* is the value of the game by the Theorem 1.3.7 or property 1 of (1.3.1).

Invertible Matrix Games (cont'd)

- Remarks
  - This method will work if the formulas we get for *X* and *Y* end up satisfying the condition that they are strategies. If either *X* or *Y* has a negative component, then it fails.
  - The strategies do not have to be completely mixed as we assumed from the beginning, only bona fide strategies.
  - In order to guarantee that the value of a game is not zero, we may add a constant to every element of *A* that is large enough to make all the numbers of the matrix positive.
    - Since $v(A + b) = v(A) + b$, where *b* is the constant added to every element, we can find the original $v(A)$ by subtracting *b.*
    - The optimal mixed strategies are not affected by doing that.

Example 2.2

- Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix}.$$

  - The matrix doesn't have an inverse because the determinant of *A* is 0.
  - Add 5 and get the inverse given by *B*

$$A + 5 = \begin{bmatrix} 5 & 6 & 3 \\ 6 & 3 & 8 \\ 3 & 8 & 1 \end{bmatrix} \qquad B = \frac{1}{80}\begin{bmatrix} 61 & -18 & -39 \\ -18 & 4 & 22 \\ -39 & 22 & 21 \end{bmatrix}.$$

  - Calculate using the formulas $v = 1/(J_3 B J_3^T) = 5$, and

$$X = v(J_3 B) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \quad \text{and} \quad Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right).$$

  - The value of our original game is $v(A) = v - 5 = 0.$

Example 2.3

- Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 7 & 2 & 2 \\ 5 & 2 & 8 \end{bmatrix}.$$

  - Then it is immediate that $v^- = v^+ = 2$ and there is a pure saddle $X^* = (0, 0, 1), Y^* = (0, 1, 0)$.
  - If we try to use Theorem 2.2.1, we have $\det(A) = 10$, and

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 6 & -2 & -1 \\ -23 & 11 & 3 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Example 2.3 (cont'd)

If we use the formulas of the theorem, we get

$$v = -1, \ X = v(A)(J_3 A^{-1}) = \left(3, -\frac{3}{2}, -\frac{1}{2}\right),$$

and

$$Y = v(A^{-1} J_3^T)^T = \left(-\frac{3}{5}, \frac{9}{5}, -\frac{1}{5}\right).$$

– Obviously these are completely messed up (i.e., wrong). The problem is that the components of *X* and *Y* are not nonnegative even though they do sum to 1.

Example 2.3 (cont’d)

- Maple commands to work out the invertible matrix game.

```
> restart:with(LinearAlgebra):
> A:=Matrix([[0 ,1 ,-2 ],[-1 ,-2 ,3],[2 ,-3 ,-4 ]]);
> Determinant(A);
> A:=MatrixAdd( ConstantMatrix(-1,3,3), A );
> Determinant(A);
> B:=A^(-1);
> J:=Vector[row]([1 ,1 ,1 ]);
> J.B.Transpose(J);
> v:=1/(J.B.Transpose(J));
> X:=v*(J.B);
> Y:=v*(B.Transpose(J));
```

Completely Mixed Games

- **Definition 2.2.2** *A game is* **completely mixed** *if every saddle point consisting of strategies* $X = (x_1, \ldots, x_n) \in S_n$, $Y = (y_1, \ldots, y_m) \in S_m$ *satisfies the property* $x_i > 0, i = 1, 2, \ldots, n$ *and* $y_j > 0, j = 1, 2, \ldots, m$. *So, every row and every column is used with positive probability.*
  - There is only one saddle point in a completely mixed game.
  - If you know that $v(A) \neq 0$, then the game matrix *A* must have an inverse, A^{-1}. The formulas for the value and the saddle

    $$v(A) = \frac{1}{J_n A^{-1} J_n^T}, \ X^* = v(A)\ J_n A^{-1}, \quad \text{and} \ Y^{*T} = v(A) A^{-1} J_n^T$$

    from Theorem 2.2.1 will give the completely mixed saddle.

Example 2.4

- **Hide and Seek.** Suppose that we have $a_1 > a_2 > \cdots > a_n > 0.$ The game matrix is

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

  - We think that the game is completely mixed.
  - Since $v^- = 0$ and $v^+ = a_n,$ the value of the game satisfies $0 \leq v(A) \leq a_n.$
  - Choosing $X = (1/n, \ldots, 1/n)$ we see that $\min_Y XAY^T = a_n/n > 0$ so that $v(A) > 0.$

Example 2.4 (cont'd)

– It is also easy to see that

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_n} \end{bmatrix}.$$

– Then, we may calculate from Theorem 2.2.1 that

$$v(A) = \frac{1}{J_n A^{-1} J_n^T} = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}},$$

$$X^* = v(A)\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right) = Y^*.$$

Example 2.4 (cont'd)

– Notice that for any $i = 1, 2, \ldots, n$, we obtain

$$1 < a_i \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right),$$

so that $v(A) < \min(a_1, a_2, \ldots, a_n) = a_n$.

Symmetric Games

Skew Symmetric

- **Skew symmetric**
  - In symmetric games, the players can use the exact same set of strategies, and the two players can switch roles. Such games can be identified by the rule that $A = -A^T$.
    - If A is the payoff matrix to player I, then the entries represent the payoffs to player I and the negative of the entries, or –A represent the payoffs to player II.
    - This means that from player II's perspective, the game matrix must be $-A^T$
    - Because A is the payoff matrix to player I and $-A^T$ is the payoff to player II. Set the payoffs to be the same and get $A = -A^T$.

Symmetric Games

- **Theorem 2.3.1** *For any skew symmetric game $v(A) = 0$ and if X^* is optimal for player I, then it is also optimal for player II.*

Proof. Let X be any strategy for I. Then

$$E(X,X) = X\ A\ X^T = -X\ A^T\ X^T = -(X\ A\ X^T)^T = -X\ A\ X^T = -E(X,X).$$

Therefore $E(X,X) = 0$ and any strategy played against itself has zero payoff.

Let (X^*, Y^*) be a saddle point for the game so that $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$, for all strategies (X, Y). Then for any (X, Y), we have

$$E(X,Y) = X\ A\ Y^T = -X\ A^T\ Y^T = -(X\ A^T\ Y^T)^T = -Y\ A\ X^T = -E(Y,X).$$

Symmetric Games (cont'd)

Hence, from the saddle point definition, we obtain

$$E(X,Y^*) = -E(Y^*,X) \leq E(X^*,Y^*) = -E(Y^*,X^*) \leq E(X^*,Y) = -E(Y,X^*).$$

Then

$$-E(Y^*,X) \leq -E(Y^*,X^*) \leq -E(Y,X^*) \Longrightarrow$$
$$\boxed{E(Y^*,X) \geq E(Y^*,X^*) \geq E(Y,X^*).}$$

But this says that Y^* is optimal for player I and X^* is optimal for player II and also that $E(X^*,Y^*) = -E(Y^*,X^*) \Longrightarrow v(A) = 0$.

Example 2.5

- **General Solution of 3 x 3 Symmetric Games.**
  - For any 3x3 symmetric game we must have

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Any of the following conditions gives a pure saddle point:

1. $a \geq 0, b \geq 0 \Longrightarrow$ saddle at $(1, 1)$ position,
2. $a \leq 0, c \geq 0 \Longrightarrow$ saddle at $(2, 2)$ position,
3. $b \leq 0, c \leq 0 \Longrightarrow$ saddle at $(3, 3)$ position.

- Here's why. Let's assume that $a \leq 0, c \geq 0$. In this case if $b \leq 0$ we get $v^- = \max\{\min\{a, b\}, 0, -c\} = 0$ and $v^+ = \min\{\max\{-a, -b\}, 0, c\} = 0$, so there is a saddle in pure strategies at $(2, 2)$. All cases are treated similarly. To have a mixed strategy, all three of these must fail.

Example 2.5 (cont'd)

– We next solve the case $a > 0, b < 0, c > 0$ so there is no pure saddle and we look for the mixed strategies.
Let player I's optimal strategy be $X^* = (x_1, x_2, x_3)$. Then

$$\begin{aligned} E(X^*, 1) &= -ax_2 - bx_3 \geq 0 = v(A) \\ E(X^*, 2) &= ax_1 - cx_3 \geq 0 \\ E(X^*, 3) &= bx_1 + cx_2 \geq 0 . \end{aligned}$$

Each one is nonnegative since $E(X^*, Y) \geq 0 = v(A)$, for all Y. Now, since $a > 0, b < 0, c > 0$ we get

$$\frac{x_3}{a} \geq \frac{x_2}{-b}, \qquad \frac{x_1}{c} \geq \frac{x_3}{a}, \qquad \frac{x_2}{-b} \geq \frac{x_1}{c}$$

so

$$\frac{x_3}{a} \geq \frac{x_2}{-b} \geq \frac{x_1}{c} \geq \frac{x_3}{a} .$$

Example 2.5 (cont'd)

Let $x_3 = a\,\lambda, x_2 = -b\,\lambda, x_1 = c\,\lambda.$ Since they must sum to one, $\lambda = 1/(a-b+c)$. We have found the optimal strategies $X^* = Y^* = (a\,\lambda, -b\,\lambda, c\,\lambda)$. The value of the game, of course is zero.

– For example, the matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$$

is skew symmetric and does not have a saddle point in pure strategies. Using the formulas in the case $a > 0, b < 0, c > 0$, we get $X^* = (\frac{3}{8}, \frac{3}{8}, \frac{2}{8}) = Y^*$, and $v(A) = 0$.

Example 2.6

- Two companies will introduce a number of new products that are essentially equivalent. They will introduce one or two products but they each must also guess how many products their opponent will introduce.

  The payoff is determined by whoever <u>introduces more products</u> and <u>guesses the correct introduction</u> of new products by the opponent. If they introduce the same number of products and guess the correct number the opponent will introduce, the payoff is zero.

Example 2.6 (cont'd)

Here is the payoff matrix to player I and strategies represent (introduce, guess).

player I / player II	(1, 1)	(1, 2)	(2, 1)	(2, 2)
(1, 1)	0	1	−1	−1
(1, 2)	−1	0	−2	−1
(2, 1)	1	2	0	1
(2, 2)	1	1	−1	0

Example 2.6 (cont'd)

– This game is symmetric.
– Drop the first row and the first column by dominance and are left with the following symmetric game:

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

– Since $a \leq 0, c \geq 0$, we have a saddle point at position $(2, 2)$. We have $v = 0, X^* = (0, 0, 1, 0) = Y^*$.
– Each company should introduce two new products and guess that the opponent will introduce one.

Example 2.7

- A game with the object of trying to find the optimal point at which to shoot. Each pistol has exactly one bullet. They will face each other starting at 10 paces apart and walk toward each other, each deciding when to shoot. A player does not know whether the opponent has taken the shot.
  - In a silent duel a player does not know whether the opponent has taken the shot.
  - In a noise duel, the players know when a shot is taken.
  - This game is assumed to be a silent duel because it is more interesting.

Example 2.7 (Cont'd)

- Suppose that they can choose to fire at 10 paces, 6 paces, or 2 paces. Suppose also that the probability that a shot hits and kills the opponent is 0.2 at 10 paces, 0.4 at 6 paces, and 1.0 at 2 paces. An opponent who is hit is assumed killed.

Example 2.7 (cont'd)

The row and column players are Burr(B) and Hamilton (H).

The player strategies consist of the pace distance at which to take the shot. The payoff to both players is +1 if they kill their opponent, -1 if they are killed, and 0 if they both survive.

B/H	10	6	2
10	0	−0.12	−0.6
6	0.12	0	−0.2
2	0.6	0.2	0

– $(+1) \cdot Prob(\text{H misses at } 10) \cdot Prob(\text{Kill H at } 6)$
$$+ (-1) \cdot Prob(\text{Killed by H at } 10) = 0.8 \cdot 0.4 - 0.2 = 0.12.$$

– This is a symmetric game with a pure saddle at position (3, 3) in the matrix, so that X^* = (0,0,1) and Y^* = (0,0,1). Both players should wait until the probability of a kill is certain.

Example 2.7 (cont'd)

- Modify the rules of the game by assuming that the players will be penalized if they wait until 2 paces to shoot.

B/H	10	6	2
10	0	−0.12	1
6	0.12	0	−0.2
2	−1	0.2	0

  - This is still a symmetric game with skew symmetric matrix, so the value is still zero and the optimal strategies are the same for both Burr and Hamilton.

Example 2.7 (cont'd)

– Find the optimal strategy for Burr in the following procedure:

$$E(X^*, 1) = 0.12\, x_2 - 1 \cdot x_3 \geq 0$$
$$E(X^*, 2) = -0.12\, x_1 + 0.2\, x_3 \geq 0$$
$$E(X^*, 3) = x_1 - 0.2\, x_2 \geq 0$$

These give us

$$x_2 \geq \frac{x_3}{0.12}, \; \frac{x_3}{0.12} \geq \frac{x_1}{0.2}, \; \text{and} \; \frac{x_1}{0.2} \geq x_2,$$

$$\Rightarrow \qquad x_1 = \frac{0.2}{0.12 + 1 + 0.2} = \frac{0.2}{1.32}, \; x_2 = \frac{1}{1.32}, \; x_3 = \frac{0.12}{1.32}$$

or $x_1 = 0.15, x_2 = 0.76, x_3 = 0.09$

So each player will shoot with probability 0.76 at 6 paces.

Matrix Games and Linear Programming

Linear Programming

- Linear programming is an area of optimization theory that is used to find the minimum (or maximum) of a linear function of many variables subject to a collection of linear constraints on the variables.
  - **Simplex method** will quickly solve very large problems formulated as linear programs.
  - Using linear programming, we can find the value and optimal strategies for a matrix game of any size without any special theorems or techniques.

- Two ways to set up a game as a linear program
  - To do by hand since it is in **standard form** (method 1).
  - Use **Maple** and involve no conceptual transformations (method 2).

Standard Form

- A linear programming problem is a problem of the standard form (called the **primal program):**

$$\text{Minimize } z = \mathbf{c} \cdot \mathbf{x}$$

$$\text{subject to } \mathbf{x}\, A \geq \mathbf{b}, \mathbf{x} \geq 0,$$

where $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $A_{n\times m}$ is an $n \times m$ matrix, and $\mathbf{b} = (b_1, \ldots, b_m)$.

  - The primal problem seeks to minimize a **linear objective function**, $z(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$, over a set of constraints (viz., $\mathbf{x} \cdot A \geq \mathbf{b}$) that are also linear.
  - The minimum and maximum of a linear function over a variable that is in a convex set must occur on the boundary of the convex set.
  - The method for solving a linear program is to go through the extreme points to find the best one. That is essentially the simplex method.

Standard Form (cont'd)

- For any primal there is a related linear program called the **dual program:**

$$\text{Maximize } w = \mathbf{y}\ \mathbf{b}^T$$
$$\text{subject to } A\ \mathbf{y}^T \leq \mathbf{c}^T, \mathbf{y} \geq 0.$$

  - **Duality theorem** states that if we solve the primal problem and obtain the optimal objective $z = z^*$, and solve the dual obtaining the optimal $w = w^*$, then $\underline{z^* = w^*}$.
  - The two objectives in the primal and the dual will give us the value of the game.

Setting up the Linear Program: Method 1

- Procedure
  - Assume that $v(A) > 0$. Now consider the properties of optimal strategies (1.3.1). Player I looks for a mixed strategy $X = (x_1, \ldots, x_n)$ so that

$$E(X, j) = XA_j = x_1\, a_{1j} + \cdots + x_n\, a_{nj} \geq v, \; 1 \leq j \leq m. \qquad (2.4.1)$$

  - It is player I's objective to get the largest value possible.
  - Change variables by setting

$$p_i = \frac{x_i}{v}, \;\; 1 \leq i \leq n, \;\; \mathbf{p} = (p_1, \ldots, p_n).$$

This is where we need $v > 0$. Then $\sum x_i = 1$ implies that

$$\sum_{i=1}^{n} p_i = \frac{1}{v}.$$

Setting up the Linear Program: Method 1 (cont'd)

- Thus maximizing v is the same as **minimizing** $\sum p_i = 1/v$.
- Divide the inequalities (2.4.1) by *v* and switch to the new variables, we get the set of constraints

$$\frac{x_1}{v} a_{1j} + \cdots + \frac{x_n}{v} a_{nj} = p_1 a_{1j} + \cdots + p_n a_{nj} \geq 1, \ 1 \leq j \leq m.$$

- So the linear programming

$$\text{Player I's program} = \begin{cases} \text{Minimize } z_{\text{I}} = \mathbf{p}\, J_n^T = \displaystyle\sum_{i=1}^{n} p_i, & J_n = (1, 1, \ldots, 1) \\ \text{subject to: } \mathbf{p}\, A \geq J_m, & \mathbf{p} \geq 0. \end{cases}$$

Setting up the Linear Program: Method 1 (cont'd)

– Notice that the constraint of the game $\sum_i x_i = 1$ is used to get the objective function! It is not one of the constraints of the linear program. The set of constraints is

$$\mathbf{p}\, A \geq J_m \iff \mathbf{p} \cdot A_j \geq 1, j = 1, \ldots, m.$$

Also $\mathbf{p} \geq 0$ means $p_i \geq 0, i = 1, \ldots, n$.

– Unwinding the formulation back to our original variables, we find the optimal strategy *X* for player I and the value of the game as follows:

(the minimum objective z_{I}, labeled z_{I}^*)

$$\boxed{value(A) = \frac{1}{\sum_{i=1}^{n} p_i} = \frac{1}{z_{\mathrm{I}}^*} \quad \text{and} \quad x_i = p_i\, value(A).}$$

Setting up the Linear Program: Method 1 (cont'd)

- Similarly, for player II we obtain

$$\text{Player II's program} = \begin{cases} \text{Maximize } z_{\text{II}} = \mathbf{q}\, J_m^T, & J_m = (1, 1, \ldots, 1), \\ \text{subject to: } A\,\mathbf{q}^T \le J_n^T & , \mathbf{q} \ge 0. \end{cases}$$

  and

$$\boxed{value(A) = \frac{1}{\sum_{j=1}^{m} q_j} = \frac{1}{z_{\text{II}}^*} \quad \text{and} \quad y_j = q_j\ value(A).}$$

  – Player II's problem is the **dual** of player I's.

Duality Theorem

- **Theorem 2.4.1 (Duality Theorem)** *If one of the pair of linear programs (primal and dual) has a solution, then so does the other. If there is at least one feasible solution (i.e., a vector that solves all the constraints so the constraint set is nonempty), then there is an optimal feasible solution for both, and their values, i.e. the objectives, are equal.*
  - This means that in a game we are guaranteed that $z_{\mathrm{I}}^* = z_{\mathrm{II}}^*$ and so the values given by player I's program will be the same as that given by player II's program.
  - If you had to add a number to the matrix to guarantee that v>0, then you have to subtract that number from z_{I}^*, and z_{II}^*, in order to get the value of the original game with the starting matrix A.

Example 2.8

- Use the linear programming method to find a solution of the game with matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -3 & -1 \\ 0 & 2 & -3 \end{bmatrix}.$$

  - A add 4 to get

$$A' = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 1 & 3 \\ 4 & 6 & 1 \end{bmatrix}.$$

Example 2.8 (cont'd)

– Setting $p_i = \dfrac{x_i}{v}$, and get

$$\text{Player I's program} = \begin{cases} \text{Minimize } z_{\mathrm{I}} = p_1 + p_2 + p_3 \quad \left(= \dfrac{1}{v}\right) \\ \text{subject to} \\ 2p_1 + 6p_2 + 4p_3 \geq 1 \\ 5p_1 + p_2 + 6p_3 \geq 1 \\ 4p_1 + 3p_2 + p_3 \geq 1 \\ p_i \geq 0 \qquad\qquad i = 1, 2, 3. \end{cases}$$

– After finding the p_i's, we will set

$$v = \frac{1}{z_{\mathrm{I}}^*} = \frac{1}{p_1 + p_2 + p_3}$$

and then the original value of A is the value of A' subtracts 4.

– $x_i = v\, p_i$ will give the optimal strategy for player I.

Example 2.8 (cont'd)

- Similarly, setting $q_j = (y_j/v)$, player II's problem is

$$
\text{Player II's program} = \begin{cases} \text{Maximize } z_{\text{II}} = q_1 + q_2 + q_3 \quad \left(= \dfrac{1}{v}\right) \\ \text{subject to} \\ 2q_1 + 5q_2 + 4q_3 \leq 1 \\ 6q_1 + q_2 + 3q_3 \leq 1 \\ 4q_1 + 6q_2 + q_3 \leq 1 \\ q_j \geq 0 \qquad\qquad j = 1, 2, 3. \end{cases}
$$

Example 2.8 (cont'd)

- The simplex method is part of all standard Maple and Mathematical software, so we will solve the linear programs using Maple. For player I we use the Maple commands:

```
> with(simplex):
> cnsts:={2*p[1]+6*p[2]+4*p[3] >=1,
              5*p[1]+p[2]+6*p[3] >=1,4*p[1]+3*p[2]+p[3] >=1};
> obj:=p[1]+p[2]+p[3];
> minimize(obj,cnsts,NONNEGATIVE);
```

  - We get the solutions

$$p_1 = \frac{21}{124},\ p_2 = \frac{13}{124},\ p_3 = \frac{1}{124} \text{ and } \quad p_1 + p_2 + p_3 = \frac{35}{124}$$

$$v(A') = \frac{124}{35}, \quad v(A) = \tfrac{124}{35} - 4 = -\tfrac{16}{35},\ X^* = (\tfrac{21}{35}, \tfrac{13}{35}, \tfrac{1}{35}).$$

Example 2.8 (cont'd)

- We may also use the `Optimization` package in Maple to solve player I's program:

```
> with(Optimization):
> cnsts:={2*p[1]+6*p[2]+4*p[3] >=1,
          5*p[1]+p[2]+6*p[3] >=1,
          4*p[1]+3*p[2]+p[3] >=1};
>obj:=p[1]+p[2]+p[3];
>Minimize(obj,cnsts,assume=nonnegative);
```

  - We get the solutions (in floating-point form)

$$p[1] = 0.169, \quad p[2] = 0.1048, \quad p[3] = 0.00806$$

$$p_1 + p_2 + p_3 = 0.28186$$

$$v = 1/0.28186 - 4 = -0.457\,.$$

Example 2.8 (cont'd)

- Solve the program for player II by using the Maple commands:

```
> with(simplex):
> cnsts:={2*q[1]+5*q[2]+4*q[3]<=1,
          6*q[1]+q[2]+3*q[3]<=1,
          4*q[1]+6*q[2]+q[3]<=1};
> obj:=q[1]+q[2]+q[3];
> maximize(obj,cnsts,NONNEGATIVE);
```

  - We get the solutions

$$q_1 = \frac{13}{124}, q_2 = \frac{10}{124}, q_3 = \frac{12}{124}, \; q_1 + q_2 + q_3 = 1/v = \frac{35}{124},$$

$$v(A') = \frac{124}{35}, \quad v(A) = \frac{124}{35} - 4 = -\frac{16}{35},$$

$$Y^* = \frac{124}{35}\left(\frac{13}{124}, \frac{10}{124}, \frac{12}{124}\right) = \left(\frac{13}{35}, \frac{10}{35}, \frac{12}{35}\right).$$

Example 2.8 (cont'd)

- Remark
  - The linear programs for each player are the **duals** of each other. Precisely, for player I the problem is

$$\text{Minimize } \mathbf{c} \cdot \mathbf{p}, \quad \mathbf{c} = (1,1,1) \text{ subject to } A^T \mathbf{p} \geq \mathbf{b}, \mathbf{p} \geq 0,$$

where $\mathbf{b} = (1,1,1)$.

  - The dual of this is the linear programming problem for player II:

$$\text{Maximize } \mathbf{b} \cdot \mathbf{q} \quad \text{subject to } A\mathbf{q} \leq \mathbf{c}, \ \mathbf{q} \geq 0.$$

Example 2.9

- **A Nonsymmetric Noisy Duel.** We consider a nonsymmetric duel at which the two players may shoot at paces (10,6,2) with accuracies (0.2,0.4,1.0) each.
  - Rules
    - If only player I survives, then player I receives payoff a.
    - If only player II survives, player I gets payoff $b < a$. This assumes that the survival of player II is less important than the survival of player I.
    - If both players survive, they each receive payoff zero.
    - If neither player survives, player I receives payoff g.

Example 2.9 (cont'd)

– Expected payoff matrix for player I (take $a = 1, b = \frac{1}{2}, g = 0$):

I/II	(0.2, 10)	(0.4, 6)	(1, 2)
(0.2, 10)	0.24	0.6	0.6
(0.4, 6)	0.9	0.36	0.70
(1.0, 2)	0.9	0.8	0

The pure strategies are labeled with the two components (accuracy,paces).

Example 2.9 (cont'd)

- Solution
  - From the general formula,

$$E((x,i),(y,j)) = \begin{cases} ax + b(1-x) & \text{if } x < y; \\ ax + bx + (g-a-b)x^2 & \text{if } x = y; \\ a(1-y) + by & \text{if } x > y. \end{cases}$$

and

$$\begin{aligned} E((x,i),(x,i)) &= aProb(\text{II misses})Prob(\text{I hits}) \\ &+ bProb(\text{I misses})Prob(\text{II hits}) + gProb(\text{I hits})Prob(\text{II hits}) \\ &= a(1-x)x + b(1-x)x + g(x \cdot x) \end{aligned}$$

.

Example 2.9 (cont'd)

– Use the Maple commands to calculate the constraints:

```
> with(LinearAlgebra):
> R:=Matrix([[0.24,0.6,0.6],[0.9,0.36,0.70],[0.9,0.8,0]]);
> with(Optimization): P:=Vector(3,symbol=p);
> PC:=Transpose(P).R;
> Xcnst:={seq(PC[i]>=1,i=1..3)};
> Xobj:=add(p[i],i=1..3);
> Z:=Minimize(Xobj,Xcnst,assume=nonnegative);
> v:=evalf(1/Z[1]); for i from 1 to 3 do evalf(v*Z[2,i]) end do;
```

Alternatively, use the simplex package:

```
> with(simplex):Z:=minimize(Xobj,Xcnst,NONNEGATIVE);
```

Example 2.9 (cont'd)

– Results

$$Z = [1.821, [p_1 = 0.968, p_2 = 0.599, p_3 = 0.254]]$$

$$v(A) = 1/1.821 = 0.549$$

$$X^* = (0.532, 0.329, 0.140).$$

Similar for player II:

$$Y^* = (0.141, 0.527, 0.331).$$

A Direct Formulation Without Transforming: Method 2

- Problems

$$\begin{cases} \text{Maximize } v \\ \text{Subject to } \displaystyle\sum_{i=1}^{n} a_{ij}x_i^* = X^*A_j = E(X^*, j) \geq v, \; j = 1, \ldots, m, \\ \displaystyle\sum_{i=1}^{n} x_i^* = 1, \; x_i \geq 0, \; i = 1, \ldots, n. \end{cases}$$

$$\begin{cases} \text{Minimize } v \\ \text{Subject to } \displaystyle\sum_{j=1}^{m} a_{ij}y_j^* = {}_iAY^{*T} = E(i, Y^*) \leq v, \; i = 1, \ldots, n, \\ \displaystyle\sum_{j=1}^{m} y_j^* = 1, \; y_j \geq 0, \; j = 1, \ldots, m. \end{cases}$$

A Direct Formulation Without Transforming: Method 2 (cont'd)

- We can solve these programs directly without changing to new variables.
- Since we don't have to divide by v in the conversion, we don't need to ensure that $v > 0$, so we can avoid having to add a constant to A.
- This formulation is much easier to set up in **Maple**.
- But, if you ever have to solve a game by hand using the simplex method, the first method is much easier.

Example 2.10

- Solve by the linear programming method with the second formulation the game with the skew symmetric matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

  – Setup for solving this using Maple.

```
> with(LinearAlgebra):with(simplex):
>#Enter the matrix of the game here,row by row:
> A:=Matrix([[0,-1,1],[1,0,-1],[-1,1,0]]);
>#The row player's Linear Programming problem:
> X:=Vector(3,symbol= x);
#Defines X as a column vector with 3 components
```

Example 2.10 (cont'd)

```
> B:=Transpose(X).A;
# Used to calculate the constraints; B is a vector.
> cnstx:={seq(B[i] >=v,i=1..3),add(x[i],i=1..3)=1};
#The components of B must be >=v and the
#components of X must sum to 1.
> maximize(v,cnstx,NONNEGATIVE);
#player I wants v as large as possible
#Hitting enter will give X=(1/3,1/3,1/3) and v=0.
>#Column players Linear programming problem:
> Y:=<y[1],y[2],y[3]>;#Another way to set up the vector for Y.
> B:=A.Y;
> cnsty:={seq(B[j]<=w,j=1..3),add(y[j],j=1..3)=1};
>minimize(w,cnsty,NONNEGATIVE);
#Again, hitting enter gives Y=(1/3,1/3,1/3) and w=0.
```

– Maple gives us the optimal strategies $X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = Y^*$.

Example 2.10 (cont'd)

- Remark
  - In the Maple statement `maximize(v,cnstx,NONNEGATIVE)` the term `NONNEGATIVE` means that Maple is trying to solve this problem by looking for **all variables** ≥ 0.
    - If it happens that the actual value is < 0, then Maple will not give you the solution.
  - You can do either of two things to fix this:

1. Drop the `NONNEGATIVE` word and change `cnstx` to

```
> cnstx:={seq(B[i] >=v,i=1..3),seq(x[i] >= 0,i=1..3),
                                     add(x[i],i=1..3)=1};
```

which puts the nonnegativity constraints of the strategy variables directly into `cnstx`. You have to do the same for `cnsty`.

2. Add a large enough constant to the game matrix A to make sure that $v(A) > 0$.

Example 2.11

- **Colonel Blotto Games.** Suppose that there are two opponents (players), which we call Red and Blue. Blue controls four regiments, and Red controls three. There are two targets of interest, say, *A* and *B.*

  The rules of the game say that the player <u>who sends the most regiments to a target will win one point for the win</u> and <u>one point for every regiment captured at that target</u>. A tie, in which Red and Blue send the same number of regiments to a target, gives a zero payoff.

Example 2.11 (cont'd)

The possible strategies consist of the number of regiments to send to *A* and *B.* The payoff matrix to Blue is

Blue/Red	$(3,0)$	$(0,3)$	$(2,1)$	$(1,2)$
$(4,0)$	4	0	2	1
$(0,4)$	0	4	1	2
$(3,1)$	1	−1	3	0
$(1,3)$	−1	1	0	3
$(2,2)$	−2	−2	2	2

– The Blue strategies (4,0) and (0,4) should be played with the same probability. The same should be true for (3,1) and (1,3) and Red. So

$$X = (x_1, x_1, x_2, x_2, x_3),\ 2x_1 + 2x_2 + x_3 = 1.$$

$$Y = (y_1, y_1, y_2, y_2),\ 2y_1 + 2y_2 = 1.$$

Example 2.11 (cont'd)

– For Red,

$$E(1, Y) = 4y_1 + 3y_2 \le v,$$
$$E(3, Y) = 3y_2 \le v, \text{ and}$$
$$E(5, Y) = -4y_1 + 4y_2 \le v.$$

$3y_2 \le 4y_1 + 3y_2 \le v$ implies $3y_2 \le v$ automatically,
so we can drop the second inequality. Since $2y_1 + 2y_2 = 1$, we have

$$4y_1 + 3y_2 = y_1 + \frac{3}{2} \le v, \quad -4y_1 + 4y_2 = -8y_1 + 2 \le v.$$

So we get $y_1 = \frac{1}{18}$, $y_2 = \frac{8}{18}$, $v = \frac{28}{18}$, and $Y^* = (\frac{1}{18}, \frac{1}{18}, \frac{8}{18}, \frac{8}{18})$.

$$(E(i, Y^*), i = 1, 2, 3, 4, 5) = \left(\frac{14}{9}, \frac{14}{9}, \frac{12}{9}, \frac{12}{9}, \frac{14}{9}\right),$$

so that $E(i, Y^*) \le \frac{14}{9}$, $i = 1, 2, \ldots, 5$.

Example 2.11 (cont'd)

– For Blue, we have $E(3, Y^*) = 3y_2 = \frac{24}{18} < \frac{28}{18}$ and, because it is a strict inequality, property 4 of the Properties (1.3.1), tells us that Blue would have 0 probability of using row 3, that is, $x_2 = 0$.

$$E(X,1) = 4x_1 - 2x_3 \geq v = \frac{28}{18},\ E(X,3) = 3x_1 + 2x_3 \geq \frac{28}{18}.$$

In addition, we have $2x_1 + x_3 = 1$.

So the solution yields $X^* = (\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9})$.

– Observations: It is optimal for the superior force (Blue) to not divide its regiments, but for the interior force to split its regiments, except for a small probability of doing the opposite.

Example 2.11 (cont'd)

- If we use Maple to solve this problem, we use the commands:

```
>with(LinearAlgebra):
>A:=Matrix([[4,0,2,1],[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>X:=Vector(5,symbol=x);
>B:=Transpose(X).A;
>cnst:={seq(B[i]>=v,i=1..4),add(x[i],i=1..5)=1};
>with(simplex):
>maximize(v,cnst,NONNEGATIVE,value);
```

The outcome is $X^* = (\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9})$ and $value = 14/9$.

Example 2.11 (cont'd)

Similarly, using the commands

```
>with(LinearAlgebra):
>A:=Matrix([[4,0,2,1],[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>Y:=Vector(4,symbol=y);
>B:=A.Y;
>cnst:={seq(B[j]<=v,j=1..5),add(y[i],j=1..4)=1};
>with(simplex):
>minimize(v,cnst,NONNEGATIVE,value);
```

results in $Y^* = (\frac{7}{90}, \frac{3}{90}, \frac{32}{90}, \frac{48}{90})$ and $v = \frac{14}{9}$.

Example 2.11 (cont'd)

- **The optimal strategy for Red is not unique, but all optimal strategies resulting in the same expected outcome.**
- **Any convex combination of the two *Y**s we found will be optimal for player II.**

$$Y = \frac{1}{2}\left(\frac{1}{18}, \frac{1}{18}, \frac{8}{18}, \frac{8}{18}\right) + \frac{1}{2}\left(\frac{7}{90}, \frac{3}{90}, \frac{32}{90}, \frac{48}{90}\right)$$
$$= \left(\frac{1}{15}, \frac{2}{45}, \frac{2}{5}, \frac{22}{45}\right)$$

$$(E(i,Y), i = 1,2,3,4,5) = (\tfrac{14}{9}, \tfrac{14}{9}, \tfrac{11}{9}, \tfrac{13}{9}, \tfrac{14}{9}).$$

- If blue deviates from the its optimal strategy $X^* = (\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9})$ by using $(\frac{3}{9}, \frac{3}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$. Then, $E(X,Y) = \frac{122}{81} < v = \frac{14}{9}$.

Example 2.11 (cont'd)

- **Remark** (Maple procedure).

```
>restart:with(LinearAlgebra):
>A:=Matrix([[4,0,2,1],[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>value:=proc(A,rows,cols)
    local X,Y,B,C,cnstx,cnsty,vI,vII,vu,vl;
    X:=Vector(rows,symbol=x): Y:=Vector(cols,symbol=y):
    B:=Transpose(X).A; C:=A.Y;
    cnsty:={seq(C[j]<=vII,j=1..rows),add(y[j],j=1..cols)=1}:
    cnstx:={seq(B[i]>=vI,i=1..cols),add(x[i],i=1..rows)=1}:
    with(simplex):
    vu:=maximize(vI,cnstx,NONNEGATIVE);
    vl:=minimize(vII,cnsty,NONNEGATIVE);
    print(vu,vl);
    end:
>value(A,5,4);
```

  - The procedure will return the value and the optimal strategies.

Linear Programming and the Simplex Method

Standard Linear Programming

- The standard linear programming problem consists of maximizing (or minimizing) a **linear function** $f : \mathbb{R}^m \to \mathbb{R}$ over a special type of convex set called a **polyhedral set** $S \subset \mathbb{R}^n$, which is a set given by a collection of linear constraints

$$S = \{\mathbf{x} \in \mathbb{R}^m \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}.$$

  – $A_{n \times m}$ is a matrix

    $\mathbf{x} \in \mathbb{R}^m$ is considered as an $m \times 1$ matrix, or vector

    $\mathbf{b} \in \mathbb{R}^n$ is considered as an $n \times 1$ column matrix.

  – The extreme points of *S* are the key.

  – An extreme point cannot be written as a convex combination of two other points of *S.*

    - If $x = \lambda x_1 + (1-\lambda)x_2$, for some $0 < \lambda < 1, x_1 \in S,\ x_2 \in S$, then $x = x_1 = x_2$.

Standard Linear Programming (cont'd)

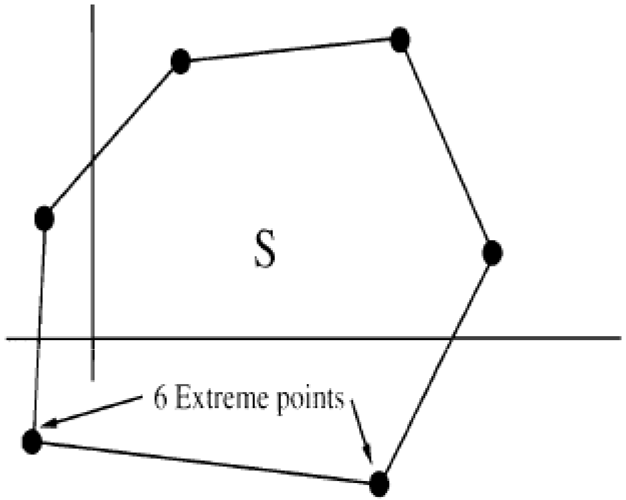

Figure 2.2 Extreme points of a convex set.

- Here are the standard linear programming problems:

$$\begin{array}{lcl} \text{Maximize } \mathbf{c}\cdot\mathbf{x} & & \text{Minimize } \mathbf{c}\cdot\mathbf{x} \\ \text{subject to} & \text{or} & \text{subject to} \\ A\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge 0 & & A\mathbf{x} \ge \mathbf{b}, \mathbf{x} \ge 0 \end{array}$$

Example 2.12

- The linear programming problem is

$$\text{Minimize } z = 2x - 4y$$
$$\text{subject to}$$
$$x + y \geq \frac{1}{2},\ x - y \leq 1,\ y + 2x \leq 3, \quad x \geq 0,\ y \geq 0.$$

  – So $c = (2, -4), b = (\frac{1}{2}, -1, -3)$ and

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & -1 \end{bmatrix}.$$

  – Graph and find that as *z* decreases, the lines go up. The furthest we can go in decreasing *z* before we leave the constraint set is at the top extreme point. That point is (0,3) and so *z* = -12.

Example 2.12 (cont'd)

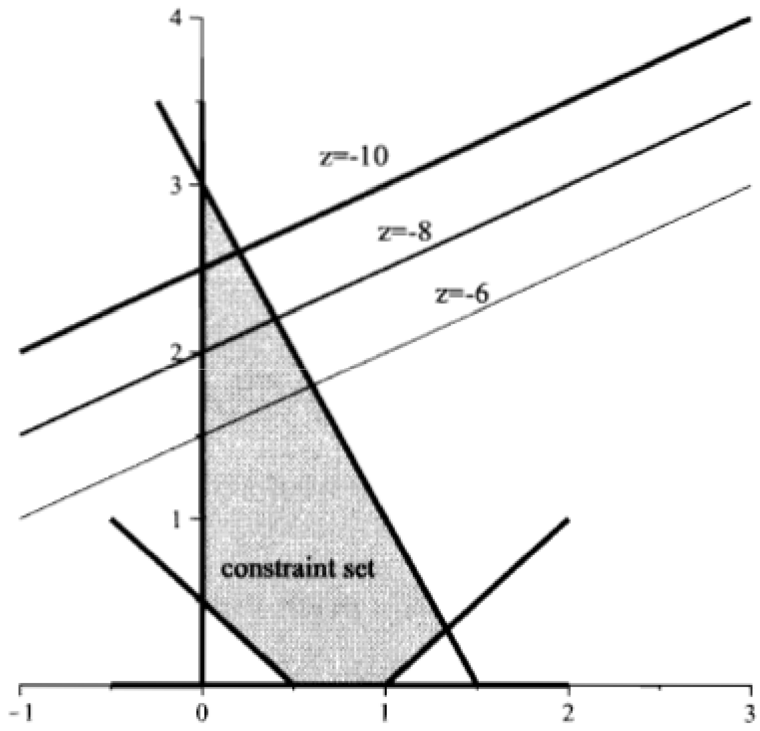

Figure 2.3 Objective Plotted over the Constraint Set

The Simplex Method

- The simplex method does not have to check all the extreme points, just the ones that improve our goal.
  - The first step in using the simplex method is to change the inequality constraints into equality constraints *A*x = b (use **slack variables**).
  - We assume that $S \neq \emptyset$. A vector $\mathbf{d} \neq 0$ is an **extreme direction** of S if and only if $A\,\mathbf{d} = 0$ and $\mathbf{d} \geq 0$.
  - The extreme directions show how to move from extreme point to extreme point in the quickest possible way, improving the objective the most.
    - If we are at an extreme point which is not our solution, then move to the next extreme point along an extreme direction.

The Simplex Method Step by Step

1. Convert the linear programming problem to a system of linear equations using **slack variables**.

2. Set up the **initial tableau**.

3. Choose a **pivot column**.

   Look at all the numbers in the bottom row, excluding the answer column. From these, choose the largest number in absolute value. The column it is in is the pivot column. If there are two possible choices, choose either one. If all the numbers in the bottom row are zero or positive, then you are done. The basic solution is the optimal solution.

The Simplex Method Step by Step (cont'd)

4. Select the **pivot** in the pivot column according to the following rules:

   (a) The pivot must always be a positive number. This rules out zeros and negative numbers.

   (b) For each positive entry b in the pivot column, excluding the bottom row, compute the ratio $\frac{a}{b}$, where a is the number in the rightmost column in that row.

   (c) Choose the smallest ratio. The corresponding number b which gave you that ratio is the **pivot**.

5. Use the pivot to clear the pivot column by row reduction. This means making the pivot element 1 and every other number in the pivot column a zero. Replace the x variable in the pivot row and column 1 by the x variable in the first row and pivot column. This results in the next tableau.

The Simplex Method Step by Step (cont'd)

6. Repeat Steps 3–5 until there are no more negative numbers in the bottom row except possibly in the answer column. Once you have done that and there are no more positive numbers in the bottom row, you can find the optimal solution easily.

7. The solution is as follows. Each variable in the first column has the value appearing in the last column. All other variables are zero. The optimal objective value is the number in the last row, last column.

- Remark
  - Player II's problem is always in standard form when we transform the game to a linear program using the first method of section 2.4. It is <u>easiest to start with player II rather than player I</u>.

A Worked Example for Simplex Method

- The game with matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix}.$$

1. The problem is

$$\text{Maximize } q := x + y + z + w$$

subject to

$$4x + y + z + 3w \leq 1, \quad 2x + 4y - 2z - w \leq 1, \quad x, y, z, w \geq 0.$$

A Worked Example for Simplex Method (cont'd)

In matrix form this

$$\text{Maximize } q := (1,1,1,1)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

subject to

$$\begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \le \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad x, y, z, w \ge 0.$$

At the end we get

$$v(A) = \frac{1}{q}, \quad Y^* = v(A)(x, y, z, w).$$

A Worked Example for Simplex Method (cont'd)

We need two slack variables to convert the two inequalities to equalities. Let's call them $s, t \geq 0$, so we have the equivalent problem

$$\text{Maximize } q = x + y + z + w + 0\,s + 0\,t$$

subject to

$$4x + y + z + 3w + s = 1,\ 2x + 4y - 2z - w + t = 1,\quad x, y, z, w, s, t \geq 0.$$

The coefficients of the objective give us the vector $\mathbf{c} = (1, 1, 1, 1, 0, 0)$.

2. Here is where we start: (initial tableau)

Variable	x	y	z	w	s	t	Answer
s	$\boxed{4}$	1	1	3	1	0	1
t	2	4	−2	−1	0	1	1
$-\mathbf{c}$	−1	−1	−1	−1	0	0	0

A Worked Example for Simplex Method (cont'd)

3. The pivot column is the column with the smallest number in the last row. Since we have several choices for the pivot column (because there are four -1s) we choose the first column arbitrarily. Notice that the last row uses $-\mathbf{c}$, not $\mathbf{c}$, because the method is designed to minimize, but we are actually maximizing here.

4. We choose the pivot in the first column by looking at the ratios $\frac{a}{b}$, where a is the number in the answer column and b is the number in the pivot column, for each row not including the bottom row. This gives $\frac{1}{4}$ and $\frac{1}{2}$ with the smaller ratio $\frac{1}{4}$. This means that the 4 in the second column is the pivot element.

A Worked Example for Simplex Method (cont'd)

5. Now we **row reduce** the tableau using the 4 in the first column until all the other numbers in that column are zero. Here is the next tableau after carrying out those row operations:

Variable	x	y	z	w	s	t	Answer
x	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$
t	0	$\boxed{\frac{7}{2}}$	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$
$-\mathbf{c}$	0	$-\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$

Notice that on the leftmost (first) column we have replaced the variable s with the variable x to indicate that the s variable is leaving (has left) and the x variable is entering.

A Worked Example for Simplex Method (cont'd)

6. We repeat the procedure of looking for a pivot column. We have a choice of choosing either column with the $\frac{3}{4}$ on the bottom. We choose the first such column because if we calculate the ratios we get $(\frac{1}{4})/(\frac{1}{4}) = 1$ and $(\frac{1}{2})/(\frac{7}{2}) = \frac{1}{7}$, and that is the smallest ratio. So the pivot element is the $\frac{7}{2}$ in the third column.

After row reducing on the pivot $\frac{7}{2}$, we get the next tableau.

Variable	x	y	z	w	s	t	Answer
x	1	0	$\boxed{\frac{3}{7}}$	$\frac{13}{14}$	$\frac{2}{7}$	$-\frac{1}{14}$	$\frac{3}{14}$
y	0	1	$-\frac{5}{7}$	$-\frac{5}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
$-\mathbf{c}$	0	0	$-\frac{9}{7}$	$-\frac{11}{14}$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{5}{14}$

The slack variable t has left and the variable y has entered.

A Worked Example for Simplex Method (cont'd)

7. The largest element in the bottom row is $-\frac{9}{7}$ so that is the pivot column. The pivot element in that column must be $\frac{3}{7}$ because it is the only positive number left in the column. So we row reduce on the $\frac{3}{7}$ pivot and finally end up with the tableau

Variable	x	y	z	w	s	t	Answer
z	$\frac{7}{3}$	0	1	$\frac{13}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$	$\frac{1}{2}$
y	$\frac{5}{3}$	1	0	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
$-\mathbf{c}$	3	0	0	2	1	0	1

.

The **maximum objective is in the lower right corner**, $q = 1$.

The **maximum is achieved** at the variables $z = \frac{1}{2}, y = \frac{1}{2}, x = 0, w = 0$, because the z and y variables are on the left column and equal the corresponding value in the answer column. The remaining variables are zero because they are not in the variable column.

We conclude that $v(A) = \dfrac{1}{q} = 1$, and $Y^* = (0, \frac{1}{2}, \frac{1}{2}, 0)$ is optimal for player II.

A Worked Example for Simplex Method (cont'd)

- To determine X^* we write down the dual of the problem for player II:

$$\text{Minimize } p = \mathbf{b}^T\mathbf{x}$$

subject to

$$A^T\mathbf{x} \geq \mathbf{c} \quad \text{and} \quad \mathbf{x} \geq 0.$$

For convenience let's match this up with player II's problem:

$$\text{Maximize } q = \mathbf{c}\cdot\mathbf{x} = (1,1,1,1)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

subject to

$$A\mathbf{x} = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 4 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \leq \mathbf{b} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad x, y, z, w \geq 0.$$

A Worked Example for Simplex Method (cont'd)

– The cost vector is replaced by the right-hand vector of the inequality constraints for player II, we replace *A* by A^T, and the original cost vector **c** becomes the inequality constraints.
– The solution of the dual is already present in the final tableau of the primal.
  - The optimal objective for the dual is the same as the primal (**duality theorem**).
  - The optimal variables are in the bottom row corresponding to the columns headed by the slack variables.
– Solutions: $v(A) = 1/q = 1$ and $X^* = (1, 0)$.
  - The 1 comes from *s* and the 0 comes from *t.*

Example 2.13

- There are two presidential candidates, Harry and Tom, who will choose which states they will visit to garner votes. Their pollsters estimate that if, for example, Tom goes to state 2 and Harry goes to state 1 , then Tom will lose 8 percentage points to Harry in that state. Suppose that there are 3 states that each candidate can select. Here is the matrix with Tom as the row player:

Tom/Harry	State 1	State 2	State 3
State 1	12	−9	14
State 2	−8	7	12
State 3	11	−10	10

Example 2.13 (cont'd)

- Use linear programming to solve this problem.
  - Step one is to set up the linear program for player II.

$$\text{Maximize } z_{\text{II}} = q_1 + q_2 + q_3$$

subject to

$$12q_1 - 9q_2 + 14q_3 \leq 1, \quad -8q_1 + 7q_2 + 12q_3 \leq 1, \quad 11q_1 - 10q_2 + 10q_3 \leq 1,$$

and $q_1, q_2, q_3 \geq 0$.

  - Set up the initial tableau.

Variables	q_1	q_2	q_3	s_1	s_2	s_3	Answer
s_1	**12**	-9	14	1	0	0	1
s_2	-8	7	12	0	1	0	1
s_3	11	-10	10	0	0	1	1
z_{II}	-1	-1	-1	0	0	0	0

Example 2.13 (cont'd)

– After pivoting on the 12 in the second column, we replace s_1 by q_1 in the first column:

Variables	q_1	q_2	q_3	s_1	s_2	s_3	Answer
q_1	1	$-\frac{3}{4}$	$\frac{7}{6}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$
s_2	0	$\boxed{1}$	$\frac{64}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{3}$
s_3	0	$-\frac{7}{4}$	$-\frac{17}{6}$	$-\frac{11}{12}$	0	1	$\frac{1}{12}$
z_{II}	0	$-\frac{7}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$

Example 2.13 (cont'd)

– Finally, we pivot on the 1 in the third column and arrive at the final tableau:

Variables	q_1	q_2	q_3	s_1	s_2	s_3	Answer
q_1	1	0	$\frac{103}{6}$	$\frac{7}{12}$	$\frac{3}{4}$	0	$\frac{4}{3}$
q_2	0	1	$\frac{64}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{3}$
s_3	0	0	$\frac{69}{2}$	$\frac{1}{4}$	$\frac{7}{4}$	1	3
z_{II}	0	0	$\frac{75}{2}$	$\frac{5}{4}$	$\frac{7}{4}$	0	3

Example 2.13 (cont'd)

– Read off the information:

$$z_{\mathrm{II}} = 3 = \frac{1}{v(A)},\ q = \left(\frac{4}{3}, \frac{5}{3}, 0\right) \implies v(A) = \frac{1}{3} \text{ and } Y^* = \left(\frac{4}{9}, \frac{5}{9}, 0\right).$$

$$p = (\tfrac{5}{4}, \tfrac{7}{4}, 0),\ \ X^* = v(A)\, p = \left(\frac{5}{12}, \frac{7}{12}, 0\right).$$

- State 3 is never to be visited by either Tom or Harry.
- Tom should visit state 1, 5 out of 12 times and state 2, 7 out of 12 times.
- Harry should visit state 1, 4 out of 9 times and state 2, 5 out of 9 times.
- Tom ends up with a net gain of $v(A)$ = 0.33%.

Example 2.13 (cont'd)

– Check these results with the Maple commands:

```
> with (LinearAlgebra):
> value:=proc(A,rows,cols)
    local X,Y,B,C,cnstx,cnsty,vI,vII,vu,vl;
    X:=Vector(rows,symbol=x):
    Y:=Vector(cols,symbol=y):
    B:=Transpose(X).A; C:=A.Y;
    cnsty:={seq(C[j]<=vII,j=1..rows),add(y[j],j=1..cols)=1}:
    cnstx:={seq(B[i]>=vI,i=1..cols),add(x[i],i=1..rows)=1}:
    with(simplex):
    vu:=maximize(vI,cnstx,NONNEGATIVE);
    vl:=minimize(vII,cnsty,NONNEGATIVE);
    print(vu,vl);
  end:

> City:=Matrix([[12,-9,14],[-8,7,12],[11,-10,10]]);
> value(City,3,3);
```

A Game Theory Model of Economic Growth

Economic Growth

- An economy has many goods (or goods and services), and there are many activities to produce the goods and services.

$$\text{Price vector } = \mathbf{p} = (p_1, p_2, \ldots, p_n).$$

$$y_j = \text{production intensity of process } j, \quad \text{and } \mathbf{y} = (y_1, \ldots, y_m).$$

  - The input process is:

$$\text{Amount of good } i = 1, 2, \ldots, n \text{ used by process } j = 1, 2, \ldots, m, \text{ is } = a_{ij} y_j \geq 0.$$

The output process is:

$$\text{Amount of good } i = 1, 2, \ldots, n, \text{ produced by process } j \text{ is } = b_{ij} y_j \geq 0.$$

Economic Growth (cont'd)

– It is assumed that the constants $a_{ij} \geq 0, b_{ij} \geq 0$, for all i, j. Set the matrices

$$A_{n\times m} = (a_{ij}), \ \text{and} \ B = (b_{ij}).$$

– For the time being we also consider two constants:

$$\rho_g = \text{ rate of growth of goods and services ,} \tag{2.6.1}$$

$$\rho_r = \text{ rate of growth of money } = \text{ risk-free interest rate.} \tag{2.6.2}$$

– Since all prices and intensities must be nonnegative

$$p_i \geq 0, \ \ i = 1, 2, \ldots, n, \quad y_j \geq 0, \ \ j = 1, 2, \ldots, m.$$

Economic Growth (cont'd)

- Assume that every row and every column of the matrices *A* and *B* has at least one positive element. This implies that

$$\sum_{i=1}^{n} a_{ij} > 0, j = 1, 2, \ldots, m \quad \text{and} \quad \sum_{j=1}^{m} b_{ij} > 0, i = 1, 2, \ldots, n.$$

- The economic meaning is that every process requires at least one good, and every good is produced by at least one process.

Input and Output Model

- Summary of the input/output model in matrix form:

1. $B\mathbf{y} \geq \rho_g A\mathbf{y}$.
2. $\mathbf{p}(\rho_g A - B)\mathbf{y} = 0$.
3. $\rho_r \mathbf{p}A \geq \mathbf{p}B$.
4. $\mathbf{p}(\rho_m A - B)\mathbf{y} = 0$.
5. $\mathbf{p}B\mathbf{y} > 0$.

Input and Output Model (cont'd)

- Condition 1 says that the output of goods must grow by a factor of ρ_g.
- Condition 2 says that if any component of $(\rho_g A - B)\mathbf{y} < 0$, then $p_i = 0$.
  - When demand is exceeded by supply, the price of that good will be zero.

  Conversely, if $p_i > 0$, the price of good i is positive, then $[(\rho_g A - B)y]_i = 0$.
  - The output of good *i* is exactly balanced by the input of good *i.*
- Conditions 3, 4 have a similar economic interpretation but for prices.
- Condition 5 is that $p_i b_{ij} y_j > 0$ for at least one i and j since the economy must produce at least one good with a positive price and with positive intensity.

A Game Theory Model of Economic Growth

- **Conclusion 1.** For an economy satisfying the assumptions of the input/output model, it must be true that the growth factor for goods must be the same as the growth rate of money $\rho_g = \rho_r$.

Proof. Using Conditions 1-5, we first have

$$\mathbf{p}(\rho_g A - B)\mathbf{y} = 0 \implies \rho_g \mathbf{p} A \mathbf{y} = \mathbf{p} B \mathbf{y} > 0$$

and

$$\mathbf{p}(\rho_m A - B)\mathbf{y} = 0 \implies \rho_r \mathbf{p} A \mathbf{y} = \mathbf{p} B \mathbf{y},$$

so that

$$\rho_g \mathbf{p} A \mathbf{y} = \rho_r \mathbf{p} A \mathbf{y} > 0.$$

Because this result is strictly positive, by dividing by $\mathbf{p}A\mathbf{y}$ we may conclude that $\rho_g = \rho_r$. □

A Game Theory Model of Economic Growth (cont'd)

- Let $\rho = \rho_g = \rho_r$. From the conditions 1-5, we have the inequalities

$$\mathbf{p}(\rho A - B) \geq 0 \geq (\rho A - B)\mathbf{y}. \tag{2.6.3}$$

  In addition

$$\mathbf{p}(\rho A - B)\mathbf{y} = 0 = \mathbf{p}(\rho A - B)\mathbf{y}. \tag{2.6.4}$$

- Consider the two-person zero sum game with matrix $\rho A - B$.

  This game has $v(\rho A - B)$, (X^*, Y^*), satisfying the saddle point condition

$$X(\rho A - B)Y^* \leq v(\rho A - B) \leq X^*(\rho A - B)Y, \text{ for all } X \in S_n, Y \in S_m.$$

- It would be nice if there were a constant $\rho = \rho_0$ so that $v(\rho_0 A - B) = 0$, because then the saddle condition becomes

$$X(\rho_0 A - B)Y^* \leq 0 \leq X^*(\rho_0 A - B)Y, \text{ for all } X \in S_n, Y \in S_m.$$

A Game Theory Model of Economic Growth (cont'd)

- In particular, for every row and column $E(i, Y^*) \leq 0 \leq E(X^*, j)$, or in matrix form
$$(\rho_0 A - B)Y^* \leq 0 \leq X^*(\rho_0 A - B),$$
which is exactly the same as (2.6.3) with $\mathbf{p}$ replaced by X^* and $\mathbf{y}$ replaced by Y^*.
- In addition, $v(\rho_0 A - B) = 0 = X^*(\rho_0 A - B)Y^*$, which is the same as (2.6.4) with $\mathbf{p}$ replaced by X^* and $\mathbf{y}$ replaced by Y^*.
- $\mathbf{p} = X^*$ and $\mathbf{y} = Y^*$ can be considered as normalized prices and intensities.
- We may assume without loss of generality from the beginning that $\sum p_i = \sum y_j = 1$.

A Game Theory Model of Economic Growth (cont'd)

- **Conclusion 2.** There is a constant $\rho_0 > 0$ (which is unique if $a_{ij} + b_{ij} > 0$) and price and intensity vectors $\mathbf{p}, \mathbf{y}$ so that $\mathbf{p}B\mathbf{y} > 0$,

$$\mathbf{p}(\rho A - B) \geq 0 \geq (\rho A - B)\mathbf{y},$$

and

$$\mathbf{p}(\rho A - B)\mathbf{y} = 0 = \mathbf{p}(\rho A - B)\mathbf{y}.$$

In other words, there is a $\rho_0 > 0$ such that $value(\rho_0 A - B) = 0$, there is a saddle point $(\mathbf{y}_{\rho_0}, \mathbf{p}_{\rho_0})$, and the saddle point satisfies $\mathbf{p}_{\rho_0} B \mathbf{y}_{\rho_0} > 0$.

A Game Theory Model of Economic Growth (cont'd)

Proof. We will show only that in fact there is ρ_0 satisfying $v(\rho_0 A - B) = 0$. If we set $f(\rho) = value(\rho A - B)$, we have $f(0) = value(-B)$. Let (X, Y) be optimal strategies for the game with matrix $-B$. Then

$$value(-B) = \min_j E_B(X, j) = \min_j \sum_{i=1}^{n} (-b_{ij}) x_i < 0$$

because we are assuming that at least one $b_{ij} > 0$ and at least one x_i must be positive. Since every row of A has at least one strictly positive element a_{ij}, it is always possible to find a large enough $\rho > 0$ so that $f(\rho) > 0$. Since $f(\rho)$ is a continuous function, the intermediate value theorem of calculus says that there must be at least one $\rho_0 > 0$ for which $f(\rho_0) = 0$. □

- There is a set of equilibrium prices, intensity levels, and growth rate of money that permits the expansion of the economy.

Example 2.14

- Consider the input/output matrices

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

  - We will find the $\rho > 0$ so that $value(\rho A - B) = 0$.
  - Notice that $a_{ij} + b_{ij} > 0$ so there is only one $\rho_0 > 0$ which will work.
  - Use Maple to find ρ. The commands we use are as follows:

```
>with(LinearAlgebra):
>value:=proc(A,rows,cols)
       local X,Y,B,C,cnstx,cnsty,vI,vII,vu,vl;
       X:=Vector(rows,symbol=x): Y:=Vector(cols,symbol=y):
       B:=Transpose(X).A;
       C:=A.Y;
```

Example 2.14 (cont'd)

```
      cnsty:={seq(C[j]<=vII,j=1..rows),add(y[j],j=1..cols)=1}:
      cnstx:={seq(B[i]>=vI,i=1..cols),add(x[i],i=1..rows)=1}:
      vu:=maximize(vI,cnstx,NONNEGATIVE);
      vl:=minimize(vII,cnsty,NONNEGATIVE);
      print(vu,vl);
 end:

> A:=a->Matrix([[-2,2*a-1,a-3],[3*a-2,a-3,2*a-1]]);
> # This is the matrix (a A-B) as a function of a>0;
> B1:=a->A(a)+ConstantMatrix(5,2,3);
> # This command adds a constant (5) to each element of A(a) so
> # that the value is not close to zero. We subtract 5 at the end to
> # get the actual value.
> value(B1(1.335),2,3);
```

Example 2.14 (cont'd)

– By plugging in various values of a, we get

$$value(B1(0),2,3)=3 \implies value(0A-B)=3-5=-2$$
$$value(B1(2),2,3)=6 \implies value(2A-B)=6-5=1$$
$$value(B1(1.5),2,3)=5.25 \implies value(1.5A-B)=5.25-5=0.25$$
$$\vdots$$
$$value(B1(1.335),2,3)=5.0025$$
$$\implies value(1.335A-B)=5.0025-5=0.0025.$$

We eventually arrive at the conclusion that when $a=1.335$, we have

$$value(B1(1.335),2,3)=value(aA-B+5)=5.0025.$$

Subtracting 5, we get $value(1.335A-B)=0$, and so $\rho_0=1.335$. The optimal strategies are $\mathbf{p}=X=(\frac{1}{2},\frac{1}{2})$, and $\mathbf{y}=Y=(0,\frac{1}{2},\frac{1}{2})$.

Newton's Method

- A sketch of the proof of a useful result to use Newton's method to calculate ρ_0.
  - To do that, we need the derivative of $f(\rho) = value(\rho A - B)$.

    Here is the derivative from the right:

$$\lim_{h \to 0+} \frac{value((\rho + h)A - B) - value(\rho A - B)}{h}$$

$$= \max_{X \in S_n(A)} \min_{Y \in S_m(A)} X(\rho A - B)Y^T,$$

where $S_n(A)$ denotes the set of strategies that are **optimal for the game with matrix *A***. Similarly, $S_m(A)$ is the set of strategies for player II that are optimal for the game with matrix *A.*

Newton's Method (cont'd)

Proof. Suppose that (X^h, Y^h) are optimal for $A_h \equiv (\rho + h)A - B$ and (X^ρ, Y^ρ) are optimal for $A_\rho \equiv \rho A - B$. Then, if we play X^h against Y^ρ, we get

$$\begin{aligned} value((\rho + h)A - B) &\leq X^h((\rho + h)A - B)Y^{\rho T} \\ &= X^h(\rho A - B)Y^{\rho T} + hX^h AY^{\rho T} \\ &\leq value(\rho A - B) + hX^h AY^{\rho T}. \end{aligned}$$

The last inequality follows from the fact that $value(\rho A - B) \geq X^h(\rho A - B)Y^{\rho T}$ because Y^ρ is optimal for $\rho A - B$. In a similar way, we can see that

$$value((\rho + h)A - B) \geq (value(\rho A - B) + hX^\rho AY^{hT}).$$

Putting them together, we have

$$value(\rho A - B) + hX^\rho AY^{hT} \leq value((\rho + h)A - B) \leq value(\rho A - B) + hX^h AY^{\rho T}.$$

Newton's Method (cont'd)

Now, divide these inequalities by $h > 0$ to get

$$X^{\rho}AY^{hT} \leq \frac{value((\rho+h)A-B) - value(\rho A - B)}{h} \leq X^{h}(\rho A - B)Y^{\rho T}.$$

Let $h \to 0+$. Since $X^h \in S_n, Y^h \in S_m$, these strategies, as a function of h, are bounded uniformly in h. Consequently, as $h \to 0$, it can be shown that $X^h \to X^* \in S_n(\rho A - B)$ and $Y^h \to Y^* \in S_m(\rho A - B)$. We conclude that

$$X^{\rho}AY^{*T} \leq \lim_{h\to 0+} \frac{value((\rho+h)A-B) - value(\rho A - B)}{h} \leq X^{*}AY^{\rho T},$$

or

$$X^{\rho}AY^{*T} \leq f'(\rho)_{+} \leq X^{*}AY^{\rho T}.$$

Newton's Method (cont'd)

Consequently, we obtain

$$
\begin{aligned}
\min_{Y \in S_n(\rho A - B)} \max_{X \in S_n(\rho A - B)} XAY^T &\leq f'(\rho)_+ \\
&\leq \min_{Y \in S_n(\rho A - B)} X^* A Y^{\rho T} \\
&\leq \max_{X \in S_n(\rho A - B)} \min_{Y \in S_n(\rho A - B)} XAY^T.
\end{aligned}
$$

Since

$$
\max_{X \in S_n(\rho A - B)} \min_{Y \in S_n(\rho A - B)} XAY^T \leq \min_{Y \in S_n(\rho A - B)} \max_{X \in S_n(\rho A - B)} XAY^T,
$$

we have shown that

$$
\begin{aligned}
f'(\rho)_+ &= \lim_{h \to 0+} \frac{value((\rho + h)A - B) - value(\rho A - B)}{h} \\
&= \max_{X \in S_n(\rho A - B)} \min_{Y \in S_n(\rho A - B)} XAY^T.
\end{aligned}
$$

□

Newton's Method (cont'd)

- Special case

A special case of this result is that for any matrix $D_{n\times m}$, we have a sort of formula for the directional derivative of $v(A)$ in the *direction* D:

$$\lim_{h\to 0+} \frac{v(A+hD)-v(A)}{h} = \max_{X\in S_n(A)} \min_{Y\in S_n(A)} XDY^T.$$

In particular, if we fix any payoff entry of A, say, a_{ij}, and take D to be the matrix consisting of all zeros except for $d_{ij}=1$, we get a formula for the partial derivative of $v(A)$ with respect to the components of A:

$$\left(\frac{\partial v(A)}{\partial a_{ij}}\right)_+ = \max_{X\in S_n(A)} \min_{Y\in S_n(A)} XDY^T = (\max_{X\in S_n(A)} x_i)(\min_{Y\in S_n(A)} y_j). \quad (2.6.5)$$